ALGEBRAIC NUMBER THEORY - BACKPAPER

JUNE 2017

Instructions:

(i) *This exam is an open sheet exam – you can keep one* hand-written *A4 sized sheet with you as a cheat sheet (with anything written on it) for your reference.*
(ii) *The questions in section 1 are to be answered in True/False, no explanation need be given. All questions are compulsory. Each question carries 1 point.*
(iii) *In section 2 answer any three questions out of the four. In section 3 answer any two questions out of the three. Each question carries 9 points.*

1.

Question 1. *Answer in True/False. Each question carries 1 point.*

(1) *Drichlet density of an infinite set of primes is always non-zero.*
(2) *Let L/K be a Galois extension of number fields. Then for any prime $\mathfrak{P}$ of L, there is a well defined Frobenius automorphism $\left[\frac{L/K}{\mathfrak{P}}\right] \in Gal(L/K)$.*
(3) *Let L_1, L_2 be Galois extensions of a field K and let $L = L_1L_2$. Then L_1L_2 is Galois and there is a natural map $Gal(L/K) \hookrightarrow Gal(L_1/K) \times Gal(L_2/K)$ which is an injection.*
(4) *Let L be a subfield of $\mathbb{C}$ which is algebraic, finite dimensional over $\mathbb{Q}$. Then for any $x \in L$, $N_{L/\mathbb{Q}}(x) = |x|$ where $|\cdot|$ denotes the usual norm in $\mathbb{C}$.*
(5) *Let $\chi : G \to \mathbb{C}^\times$ denote a non-trivial character of a finite group G. Then $\sum_{g\in G} \chi(g) = 0$ if G is abelian but not necessarily otherwise.*

2.

Answer any three of the following. Each question carries 9 points.

Question 2. *Let L be a number field and I be an integral ideal of L. Assume $[L : \mathbb{Q}] = d$.*

(1) *Show that there are d elements $a_1, a_2, ..., a_d \in I$, linearly independent over $\mathbb{Q}$, such that $I = \mathbb{Z}a_1 + \mathbb{Z}a_2 + \cdots + \mathbb{Z}a_d$.*
(2) *Show that if $I \subset J$ with J another integral ideal of L, then the number of cosets $|J/I|$ is finite.*

Question 3. *Solve the equation $x^2 - 2y^2 = 1$ in integers.*

Question 4. *A Mersenne prime is a prime of the form $2^l - 1$ where $l \in \mathbb{N}$. Show that the Drichlet density of the set of all Mersenne primes is 0.*

Question 5. *Let $K = \mathbb{Q}[\sqrt{-17}]$. For any prime $p \in \mathbb{Z}$ determine whether p is inert, totally split or ramifies in K in terms of a congruence condition on p.*

3.

Answer any two of the following. Each question carries 9 points.

Question 6. *Compute the class group of* $K = \mathbb{Q}[\sqrt{-55}]$ *as follows:*

(1) *Use Minkowski bound to show that primes* $\mathfrak{p}$ *that lie over* (2) *and* (3) *generate the class group.*
(2) *Show that if* $\mathfrak{p}_2$ *be a prime of* K *over* (2)*, then* $\mathfrak{p}_2^2 \neq 1$ *but* $\mathfrak{p}_2^4 = 1$*.*
(3) *Show that any prime of* K *over* (3) *is principal, thereby computing the class group.*

Question 7. *Let* $L_1, L_2/K$ *be a Galois extension of number fields, and let* $L = L_1L_2$ *denote the compositum. Then show that a prime* $\mathfrak{p}$ *of* K *is totally split in* L *if and only if it is totally split in* L_1 *as well as in* L_2*.*

Question 8. *Let* $\nu(n) := \begin{cases} 0 & n = 1 \\ k & n = p_1^{a_1} p_2^{a_k} \ldots p_k^{a_k} \text{ with } p_i \neq p_j \text{ for } i \neq j \end{cases}.$

Show that

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^3} = \zeta(3) \sum_{p \text{ prime}} \frac{1}{p^3}$$

where $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$ *denotes the usual zeta function.*

Remark (Minkowski Bound)**.** *For any fractional ideal* $I \subset K$*, there is an integral ideal* $J \in [I]$ *(where* $[I]$ *denotes the class of* I *in the class group) such that:*

$$N_{K/\mathbb{Q}}(J) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|\Delta(K/\mathbb{Q})|}$$

where n *is the degree of* $K/\mathbb{Q}$*, there are* $2s$ *complex embeddings of* K *in* $\mathbb{C}$*, and* $\Delta(K/\mathbb{Q})$ *is the discriminant of* K*.*